

AN EXPONENTIAL LIMIT FORMULA FOR NONLINEAR SEMIGROUPS

BY
JOEL L. MERMIN

Abstract. In recent papers, many writers have developed the theory of semigroups of operators generated by nonlinear accretive operators. In the present paper, we construct this semigroup by means of an exponential limit formula, and then use this means of obtaining the semigroup to prove an approximation theorem that is a direct generalization of the Kato-Trotter theorem for linear semigroups.

1. Introduction. In papers by Browder [1], Kato [3], and Komura [4], a theory of nonlinear semigroups has been developed in which a relatively simple sufficient condition that an operator A generate such a semigroup, namely that A be m -accretive (see §2), has been established. Once the existence of a semigroup having A as its generator is known, it follows from results of Neuberger [6], and Ôharu [7] that the semigroup is given by the exponential limit formula $\lim_{n \rightarrow \infty} (I + tA/n)^{-n}$. In the main theorem of this paper presented in §3, we show that this procedure can be reversed by proving directly that the above exponential limit is convergent for each $t \geq 0$ to an operator T_t if A is m -accretive, and the family of operators so defined is a semigroup having A as its generator. This result is close to some recent work of G. Webb [9] in which the semigroup generated by an m -accretive operator is defined by a more general product limit formula. It should be noted that the property we call “ m -accretive” has been also called “hypermaximal accretive,” and “hyper-accretive” by various authors.

In §4, we use the exponential limit formula to prove an approximation theorem for nonlinear semigroups which generalizes a linear theorem of Kato.

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2. Duality mappings and m -accretive operators. Let X be a Banach space, and X^* its dual space. For each $x \in X$ and $f \in X^*$ denote $f(x)$ by (x, f) . A duality mapping on X is a mapping $F: X \rightarrow X^*$ such that $(x, F(x)) = |x|^2 = |F(x)|^2$ for all x . In general, there will be many duality maps on X . If X^* is uniformly convex however, then there is exactly one duality mapping F and it is uniformly continuous on bounded subsets of X (see [3]). Thus if X^* is uniformly convex, we can define

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the function $\phi(R, \varepsilon) = \text{lub} \{ \delta \leq 2R \mid |x-y| < \delta \text{ and } |x| \leq R \text{ implies } |F(x) - F(y)| < \varepsilon \}$ for all $R, \varepsilon > 0$. Then ϕ is decreasing in R , increasing in ε , and

$$(2.1) \quad |x-y| < \phi(R, \varepsilon) \text{ and } |x| \leq R \text{ implies } |F(x) - F(y)| < \varepsilon.$$

From now on, we assume X^* is uniformly convex.

An operator A with domain and range in X is accretive if and only if

$$(2.2) \quad \text{Re} (Ax - Ay, F(x-y)) \geq 0 \text{ for all } x, y \in D(A) \text{ for some duality map } F.$$

A is m -accretive if A is accretive and

$$(2.3) \quad R(I+A) = X.$$

It is easy to verify that A accretive implies

$$(2.4) \quad (I + \lambda A)^{-1} \text{ is defined on } X \text{ and}$$

$$|(I + \lambda A)^{-1}|_L \leq 1 \quad \text{for all } \lambda > 0$$

where " $|\cdot|_L$ " is the Lipschitz norm of an operator. If A is m -accretive it also follows that

$$(2.5) \quad R(I + \lambda A) = X \text{ for all } \lambda > 0 \text{ (see [3], [4], [5], [7]).}$$

For A m -accretive, we set $J_\lambda = (I + \lambda A)^{-1}$, and $A_\lambda = AJ_\lambda$. It follows that

$$(2.6) \quad A_\lambda = (I - J_\lambda)/\lambda.$$

LEMMA 2.1. (a) If A is m -accretive, then for $x \in D(A)$

$$(2.7) \quad |AJ_\lambda^k x| \leq |Ax|, \quad k = 1, 2, 3, \dots, \lambda > 0.$$

(b) If A is m -accretive, and $x \in D(A)$, then

$$(2.8) \quad |J_{\lambda_1}^k x - J_{\lambda_2}^k x| \leq k \cdot |Ax| \cdot |\lambda_1 - \lambda_2|, \quad k = 1, 2, 3, \dots$$

Proof. For $x \in D(A)$,

$$\begin{aligned} |AJ_\lambda^k x| &= |A_\lambda \cdot J_\lambda^{k-1} x| = \frac{1}{\lambda} |J_\lambda^{k-1} x - J_\lambda^k x| \quad (\text{by (2.6)}) \\ &= \frac{1}{\lambda} |J_\lambda^k \cdot (I + \lambda A)x - J_\lambda^k x| \\ &\leq \frac{1}{\lambda} |(I + \lambda A)x - x| = |Ax| \quad (\text{by (2.4)}). \end{aligned}$$

Turning to (2.8), let $k=1$. If $u = J_{\lambda_1} x$, $v = J_{\lambda_2} x$, then $u + \lambda_1 Au = v + \lambda_2 Av$ and

$$\begin{aligned} |u-v|^2 &= (\lambda_2 Av - \lambda_1 Au, F(u-v)) = (\lambda_2 - \lambda_1) \text{Re} (Av, F(u-v)) \\ &\quad - \lambda_1 \text{Re} (Au - Av, F(u-v)) \leq (\lambda_2 - \lambda_1) \cdot \text{Re} (Av, F(u-v)) \\ &\leq |\lambda_2 - \lambda_1| \cdot |AJ_{\lambda_2} x| \cdot |u-v|. \end{aligned}$$

In view of (2.7), this proves (2.8) for $k=1$. One can now establish (2.8) for $k \geq 1$ by an induction argument using the fact that $|J_\lambda|_L \leq 1$.

If A is m -accretive, then the following continuity property holds for A :

(2.9) If $x_n \in D(A)$, $\lim_{n \rightarrow \infty} x_n = x$, and $|Ax_n| \leq M$ for $n=1, 2, 3, \dots$, then $x \in D(A)$ and $\text{wk} \lim_{n \rightarrow \infty} Ax_n = Ax$. (By $\text{wk} \lim_{n \rightarrow \infty} u_n = u$, we mean (u_n, f) converges to

(u, f) for all $f \in X^*$.) Finally we need the following differentiation rule. If $|u(t)|$ and $u(t)$ are differentiable at t_0 and defined in an open interval about t_0 , then

$$(2.10) \quad (d/dt)|u(t)|_{t=t_0}^2 = 2 \operatorname{Re} (u'(t_0), Fu(t_0)).$$

For the proof of (2.9) and (2.10), see [3].

3. The exponential limit formula. A (nonlinear) semigroup on a set $D \subseteq X$ is a family of operators T_t , $t \geq 0$ on D such that:

1. $D(T_t) = D$ for all $t \geq 0$,
2. $T_0 = I_D$,
3. $T_{s+t} = T_s \cdot T_t$,
4. $T_t x$ is strongly continuous in t for each $x \in D$,
5. $|T_t|_L \leq 1$.

We note that many authors include only conditions 1–3 above in the definition of a semigroup, and use “strongly continuous contraction semigroup” for an object satisfying conditions 4 and 5 as well. Given a semigroup of operators, the operator A defined by $Ax = -\operatorname{wk} \lim_{h \rightarrow 0} (T_h x - x)/h$ with domain all x for which this limit exists, is the weak infinitesimal generator of the semigroup.

If we replace weak limit by strong limit in the above, the resulting operator is the strong generator of the semigroup.

THEOREM 3.1. *If X^* is uniformly convex and A is m -accretive on X , then for $x \in \operatorname{Cl}(D(A))$*

$$\lim_{n \rightarrow \infty} (I + tA/n)^{-n} x = T_t x$$

exists for $t \geq 0$, and the family of limit operators T_t is a semigroup on $\operatorname{Cl}(D(A))$ having A as its weak generator. If X is uniformly convex as well, A is the strong generator of the semigroup. Moreover, for each $a \in D(A)$, $u(t) = T_t a$ is an almost everywhere strongly differentiable solution of

$$(3.1) \quad u'(t) = -Au(t), \quad u(0) = a.$$

Proof. We begin by constructing a sequence of approximate solutions to (3.1). For $n = 1, 2, 3, \dots$, let $J_n = (I + A/n)^{-1}$. By (2.4), $D(J_n) = X$ and $|J_n|_L \leq 1$ for all n . By (2.6) and (2.7)

$$(3.2) \quad AJ_n = n(I - J_n),$$

$$(3.3) \quad |AJ_n^k x| \leq |Ax| \text{ for } x \in D(A) \text{ and } n, k = 1, 2, 3, \dots$$

For each $a \in D$, let

$$u_n(t) = \left[J_n^{k-1} + n \left(t - \frac{k-1}{n} \right) (J_n^k - J_n^{k-1}) \right] a \quad \text{for } \frac{k-1}{n} \leq t \leq \frac{k}{n}, \quad k = 1, 2, 3, \dots$$

Then each u_n is piecewise linear with $u_n(k/n) = J_n^k a$. In particular $u_n(0) = a$ for all n . If $(k-1)/n < t < k/n$, then

$$(3.4) \quad u'_n(t) = n(J_n^k - J_n^{k-1})a = -n(I - J_n)J_n^{k-1}a = -AJ_n^k a = -u_n(k/n).$$

It follows from (3.3) and (3.4) that

$$(3.5) \quad |u'_n(t)| \leq |Aa| \text{ for almost all } t, \text{ and all } n.$$

Now let $x_{nm}(t) = u_n(t) - u_m(t)$ for $n, m \geq 1$. Then $x_{nm}(0) = 0$ and by (3.5)

$$(3.6) \quad |x_{nm}(t)| \leq 2|Aa|t \text{ for all } n, m \text{ and } t \geq 0.$$

Now each $|x_{nm}(t)|$ is Lipschitz continuous as $x_{nm}(t)$ is, so for almost all t , $|x_{nm}(t)|$ and $x_{nm}(t)$ are differentiable. Thus by (2.10) if t satisfies $(k_n - 1)/n < t < k_n/n$ for all n , k_n a nonnegative integer, then for almost all t

$$(d/dt)|x_{nm}(t)|^2 = 2 \operatorname{Re} (u'_n(t) - u'_m(t), Fx_{nm}(t)) = -2 \operatorname{Re} (AJ_n^{k_n}a - AJ_m^{k_m}a, Fx_{nm}(t)).$$

Letting $y_{nm}(t) = J_n^{k_n}a - J_m^{k_m}a$, and using the accretiveness of A , the last term is dominated by

$$2 \operatorname{Re} (AJ_n^{k_n}a - AJ_m^{k_m}a, Fy_{nm}(t) - Fx_{nm}(t)) \leq 4|Aa| |Fy_{nm}(t) - Fx_{nm}(t)|$$

by (3.3). Hence $(d/dt)|x_{nm}(t)|^2 < \delta$ almost everywhere in $[0, T]$ if

$$|y_{nm}(t) - x_{nm}(t)| < \phi(2T|Aa|, \delta/4|Aa|)$$

by (2.1) and (3.6). Since

$$|y_{nm}(t) - x_{nm}(t)| \leq \left| u_n(t) - u_n\left(\frac{k_n}{n}\right) \right| + \left| u_m(t) - u_m\left(\frac{k_m}{m}\right) \right| \leq \left(\frac{1}{n} + \frac{1}{m} \right) |Aa|,$$

this will hold if

$$n, m > \frac{2|Aa|}{\phi(2T|Aa|, \delta/4|Aa|)}.$$

Integrating and setting $\delta = \varepsilon^2/T$, we obtain

$$(3.7) \quad |x_{nm}(t)| = |u_n(t) - u_m(t)| < \varepsilon \text{ on } [0, T] \text{ if}$$

$$n, m > \frac{2|Aa|}{\phi(2T|Aa|, \varepsilon^2/4T|Aa|)}.$$

Thus $u_n(t)$ converges uniformly on each bounded interval to a continuous $u(t)$ on $[0, +\infty)$. For any t , if $(k_n - 1)/n \leq t \leq k_n/n$, then $\lim_{n \rightarrow \infty} k_n/n = t$ and by uniform convergence $u(t) = \lim_{n \rightarrow \infty} u_n(k_n/n)$ and $|Au_i(k_n/n)| = |AJ_n^{k_n}a| \leq |Aa|$. Thus by (2.9) $u(t) \in D(A)$ and $\operatorname{wk} \lim_{n \rightarrow \infty} AJ_n^{k_n}a = Au(t)$. It follows that $|Au(t)| \leq |Aa|$ for all t , so that by (2.9) $Au(t)$ is weakly continuous, and for almost all t , $\operatorname{wk} \lim_{n \rightarrow \infty} u'_n(t) = -Au(t)$. We have for any $f \in X^*$

$$(u_n(t), f) = (a, f) + \int_0^t (u'_n(s), f) ds$$

where the integrand converges boundedly almost everywhere to $-Au(s)$. Thus letting $n \rightarrow \infty$ we get

$$(u(t), f) = (a, f) - \int_0^t (Au(s), f) ds.$$

Since the integrand is continuous, $u(t)$ has the weakly continuous weak derivative $-Au(t)$ everywhere in $[0, +\infty)$. Since weak continuity implies $Au(s)$ is Bochner

integrable, we get $u(t) = a - \int_0^t Au(s) ds$ for all t and $u(t)$ has $-Au(t)$ has a strong derivative almost everywhere by a standard result about Bochner integrals (see [8]).

Now let $v_n(t) = (I + tA/n)a^{-n}$ for $a \in D(A)$. Note that $v_n(1) = u_n(1)$ for all n . Applying the convergence estimate (3.7) with $t=T=1$, but to tA in place of A we get

$$(3.8) \quad |v_n(t) - v_m(t)| < \varepsilon \quad \text{for } n, m > \frac{2t|Aa|}{\phi(2t|Aa|, \varepsilon^2/4t|Aa|)}.$$

It follows that $v_n(t)$ is uniformly convergent on any bounded interval to a limit $v(t)$. Since for $t=p/q$ rational $u_{qn}(p/q) = v_{pn}(p/q)$, in the limit $v(t) = u(t)$ for t rational. But the $v_n(t)$ are uniformly continuous by (2.8) which implies

$$|v_n(t_1) - v_n(t_2)| \leq |Aa| \cdot |t_1 - t_2|.$$

Thus v is continuous and $v(t) = u(t)$ for all t . So for any

$$a \in D(A) \quad \lim_{n \rightarrow \infty} (I + tA/n)^{-n} a = u(t)$$

where $u(t)$ is a solution of (3.1). Since $|(I + tA/n)^{-n}|_L \leq 1$, we easily obtain that $\lim_{n \rightarrow \infty} (I + tA/n)^{-n} x = T_t x$ exists for any $x \in \text{Cl}(D(A))$. Clearly $|T_t|_L \leq 1$, and this together with the continuity of $T_t a$ for $a \in D(A)$ gives continuity of $T_t x$ for all $x \in \text{Cl}(D(A))$.

Also $T_0 = I$ on D , and for any $t \geq 0$ and positive integer m

$$T_{tm} = \lim_{n \rightarrow \infty} \left(I + \frac{tm}{n} A \right)^{-n} = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A \right)^{-nm} = T_t^m.$$

It is easy to deduce from this that $T_{s+t} = T_s \cdot T_t$.

Thus T_t , $t \geq 0$ is a semigroup, and we have left to show that A is its generator. For $a \in D(A)$, $T_t a = u(t)$ satisfies (3.1) weakly at 0 so that

$$-Aa = \text{wk} \lim_{h \rightarrow 0} (T_h a - a)/h.$$

Thus $A \subseteq \hat{A}$ where \hat{A} is the weak generator of A . But \hat{A} is accretive (see [3]) so that $I + \hat{A}$ is a one-to-one extension of $I + A$ which has range X . This is possible only if $A = \hat{A}$.

Finally, suppose X is uniformly convex. To show A is the strong generator, we must show $-Aa = \lim_{h \rightarrow 0} (T_h a - a)/h$. But

$$\frac{T_h a - a}{h} = \frac{-1}{h} \int_0^h (AT_s a) ds$$

using a Bochner integral. Hence it suffices to show that $\lim_{h \rightarrow 0} AT_h a = Aa$. This limit holds weakly so that $|Aa| \leq \lim_{h \rightarrow 0} \inf |AT_h a|$ but $|AT_h a| = |Au(h)| \leq |Aa|$ for $h > 0$

$$\therefore \lim_{h \rightarrow 0} |AT_h a| = |Aa|$$

and so by uniform convexity of X $\lim_{h \rightarrow 0} AT_h a = Aa$.

4. An approximation theorem.

THEOREM 4.1. *Let $A, A_k, k=1, 2, 3, \dots$, be m -accretive operators on X where X^* is uniformly convex. Let T_t , and $T_t^k, k=1, 2, 3, \dots$, be the associated semigroups. Suppose that*

1. $\text{Cl}(D(A_k)) = \text{Cl}(D(A)), k=1, 2, 3, \dots$
2. $\lim_{k \rightarrow \infty} (I + \lambda_0 A_k)^{-1} x = (I + \lambda_0 A)^{-1} x$, for some $\lambda_0 > 0$ and all $x \in X$.

Then for all $x \in \text{Cl}(D(A))$, $\lim_{k \rightarrow \infty} T_t^k x = T_t x$, the limit being uniform on bounded t intervals for each x in $D(A)$.

We need two preliminary lemmas.

LEMMA 4.1. *Under the hypotheses of Theorem 4.1,*

$$\lim_{k \rightarrow \infty} (I + \lambda A_k)^{-n} x = (I + \lambda A)^{-n} x \text{ for all } \lambda > 0, x \in X \text{ and } n = 1, 2, 3, \dots$$

Proof. Let $n=1$, and set $R_k(\lambda) = (\lambda I + A_k)^{-1}, k=1, 2, 3, \dots$, and $R(\lambda) = (\lambda I + A)^{-1}$. Since $R(\lambda) = (I + \lambda^{-1} A)^{-1} \cdot \lambda^{-1}$, $R_k(\lambda) = (I + \lambda^{-1} A_k)^{-1} \cdot \lambda^{-1}$, it is sufficient to show that $\lim_{k \rightarrow \infty} R_k(\lambda)x = R(\lambda)x$ holds for all $\lambda > 0$ if it holds for a single $\lambda_0 > 0$. We show that if it holds for $\lambda_0 > 0$, then it holds for all λ such that $|\lambda - \lambda_0| < \lambda_0$ from which the desired conclusion easily follows.

Suppose $\lim_{k \rightarrow \infty} R_k(\lambda_0)x = R(\lambda_0)x$ for all $x \in X$. Let λ satisfy $|\lambda - \lambda_0| < \lambda_0$, and let $u_k = R_k(\lambda)x$, $u = R(\lambda)x$. Then $\lambda u_k + A_k u_k = x$, so that $(\lambda - \lambda_0)u_k + (\lambda_0 I + A_k)u_k = x$ and $u_k = R_k(\lambda_0)(x - (\lambda - \lambda_0)u_k)$. Similarly, $u = R(\lambda_0)(x - (\lambda - \lambda_0)u)$. Then

$$(4.1) \quad |u - u_k| \leq |u - R_k(\lambda_0)(x - (\lambda - \lambda_0)u)| \\ + |R_k(\lambda_0)(x - (\lambda - \lambda_0)u) - R_k(\lambda_0)(x - (\lambda - \lambda_0)u_k)|.$$

By hypothesis, the first term on the right in (4.1) tends to 0 as $k \rightarrow \infty$, and the second term is dominated by

$$|R_k(\lambda_0)|_L \cdot |\lambda - \lambda_0| \cdot |u - u_k| \leq |\lambda - \lambda_0|/|\lambda_0| \cdot |u - u_k|.$$

Setting $c = |\lambda - \lambda_0|/|\lambda_0| < 1$, it follows that $|u - u_k|(1 - c)$ is dominated by a sequence converging to 0 so

$$\lim_{k \rightarrow \infty} u_k = (\lambda I + A_k)^{-1} x = u = (\lambda I + A)^{-1} x.$$

The convergence can be extended to $n > 1$, by induction using the uniform Lipschitz continuity of the operators $(\lambda I + A_k)^{-n}$.

LEMMA 4.2. *Let $x_k \in D(A_k), k=1, 2, 3, \dots$, where $\lim_{k \rightarrow \infty} x_k = x$ and $|A_k x_k| \leq M$ for all k and some $M > 0$. Then for any fixed n , $\lim_{k \rightarrow \infty} (I + t A_k/n)^{-n} x_k = (I + t A/n)^{-n} x$ the convergence being uniform on any bounded t interval.*

Proof. The pointwise existence of the above limit follows from Lemma 4.1 since

$$\left| \left(I + \frac{t}{n} A_k \right)^{-n} x_k - \left(I + \frac{t}{n} A \right)^{-n} x \right| \leq \left| \left(I + \frac{t}{n} A_k \right)^{-n} x - \left(I + \frac{t}{n} A \right)^{-n} x \right| + \left| \left(I + \frac{t}{n} A_k \right)^{-n} x_k - \left(I + \frac{t}{n} A_k \right)^{-n} x \right|$$

where the first term tends to zero by Lemma 4.1 and the second term, dominated by $|x_k - x|$ also tends to 0 as $k \rightarrow \infty$. But in view of (2.8)

$$\left| \left(I + \frac{t_1}{n} A_k \right)^{-n} x_k - \left(I + \frac{t_2}{n} A_k \right)^{-n} x_k \right| \leq |t_1 - t_2| \cdot |A_k x_k| \leq |t_1 - t_2| \cdot M.$$

Thus the sequence $(I + tA_k/n)^{-n} x_k$ is equicontinuous in t and the convergence must be uniform in bounded t intervals.

We can now prove the theorem. Since the semigroup operators all have Lipschitz norm uniformly less than 1 it suffices to show $\lim_{k \rightarrow \infty} T_t^k x = T_t x$ uniformly for $0 \leq t \leq T$ for any $T > 0$ for all x in the dense set $D(A)$. For such an x , we have $x = (I + A)^{-1} u$ for some $u \in X$, and setting $x_k = (I + A_k)^{-1} u$, $k = 1, 2, 3, \dots$, we have $\lim_{k \rightarrow \infty} x_k = x$ and since $A_k x_k = u - x_k$, we also have $|A_k x_k| \leq M$ for all k , and some $M > 0$. Then

$$\begin{aligned} |T_t^k x - T_t x| &\leq |T_t^k x - T_t^k x_k| + \left| T_t^k x_k - \left(I + \frac{t}{n} A_k \right)^{-n} x_k \right| \\ (4.2) \quad &+ \left| \left(I + \frac{t}{n} A_k \right)^{-n} x_k - \left(I + \frac{t}{n} A \right)^{-n} x \right| \\ &+ \left| \left(I + \frac{t}{n} A \right)^{-n} x - T_t x \right|. \end{aligned}$$

Let $\varepsilon > 0$ be given and let $0 \leq t \leq T$. By (3.8) using the fact that $|A_k x_k| \leq M$, the second and fourth terms of (4.2) can be made less than $\varepsilon/4$ for all $n \geq n_0$ independent of k . The first term of (4.2) is dominated by $|x - x_k|$ and so is less than $\varepsilon/4$ for $k \geq k_0$. Finally, taking $n = n_0$, the third term is less than $\varepsilon/4$ for $k \geq k_1$ by Lemma 4.2. Thus taking $n = n_0$, and $k > k_0, k_1$, $|T_t^k x - T_t x| < \varepsilon$ for $0 \leq t \leq T$.

Finally, we remark that our two theorems can be generalized to the case where $A = A(t)$ is a family of m -accretive operators depending on t , for $t \geq 0$. If the domain D of $A(t)$ is independent of t , and $A(t)$ varies smoothly with t in a certain sense, then the solutions of $u'(t) = -A(t)u(t)$, $u(0) = a$ can be obtained by the "time dependent" exponential formula

$$\lim_{n \rightarrow \infty} \prod_{k \leq n} \left(I + \frac{t}{n} A\left(\frac{kt}{n}\right) \right)^{-1} a = M(t)a.$$

The $M(t)$, $t \geq 0$ are a family of evolution operators associated with $A(t)$, and our second theorem can be generalized as follows. If $A_k(t)$, $k = 1, 2, 3, \dots$, and $A(t)$

are families of m -accretive operators with $D(A_k(t)) = D_k$, $k = 1, 2, 3, \dots$, $D(A(t)) = D$ independent of t , and $\bar{D}_k = \bar{D} = X$ for each k , where a certain common smoothness condition in t is satisfied by all operator families, then if

$$\lim_{k \rightarrow \infty} (I + \lambda(t)A_k(t))^{-1} = (I + \lambda(t)A(t))^{-1}$$

for each $t \geq 0$ and some $\lambda(t) > 0$, then $\lim_{k \rightarrow \infty} M_k(t)x = M(t)x$ uniformly for t in any bounded interval, and any $x \in X$, where $M_k(t)$, $t \geq 0$ is the family of evolution operators associated with $A_k(t)$, $k = 1, 2, 3, \dots$. For these time dependent results, see [5].

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UNIVERSITY OF CONNECTICUT,
STORRS, CONNECTICUT 06268